Combinatorics, operations, and graded graphs

Samuele Giraudo
LIGM, Université Paris-Est Marne-la-Vallée

Visite du comité HCERES
Exposé scientifique de l’équipe COMBI

February 12, 2019
Outline

Combinatorics

Algebraic combinatorics

Operads and graded graphs
Outline

Combinatorics
Combinatorial collections

A combinatorial collection is a set $C$ endowed with a map

$$| - | : C \to \mathbb{N}$$

such that for any $n \in \mathbb{N}$, $C(n) := \{x \in C : |x| = n\}$ is finite.

For any $x \in C$, we call $|x|$ the size of $x$. 
Combinatorial collections

A combinatorial collection is a set $C$ endowed with a map

$$| - | : C \rightarrow \mathbb{N}$$

such that for any $n \in \mathbb{N}$, $C(n) := \{x \in C : |x| = n\}$ is finite.

For any $x \in C$, we call $|x|$ the size of $x$.

— Classical questions —

1. Enumerate the objects of $C$ of size $n$. 
Combinatorial collections

A combinatorial collection is a set $C$ endowed with a map

$$\mid - \mid : C \to \mathbb{N}$$

such that for any $n \in \mathbb{N}$, $C(n) := \{x \in C : \mid x \mid = n\}$ is finite.

For any $x \in C$, we call $\mid x \mid$ the size of $x$.

— Classical questions —

1. Enumerate the objects of $C$ of size $n$.

2. Generate all the objects of $C$ of size $n$. 
A combinatorial collection is a set $C$ endowed with a map

$$| - | : C \to \mathbb{N}$$

such that for any $n \in \mathbb{N}$, $C(n) := \{x \in C : |x| = n\}$ is finite.

For any $x \in C$, we call $|x|$ the size of $x$.

— Classical questions —

1. Enumerate the objects of $C$ of size $n$.
2. Generate all the objects of $C$ of size $n$.
3. Randomly generate an object of $C$ of size $n$. 
Combinatorial collections

A combinatorial collection is a set $C$ endowed with a map

$$| - | : C \to \mathbb{N}$$

such that for any $n \in \mathbb{N}$, $C(n) := \{ x \in C : |x| = n \}$ is finite.

For any $x \in C$, we call $|x|$ the size of $x$.

— Classical questions —

1. Enumerate the objects of $C$ of size $n$.
2. Generate all the objects of $C$ of size $n$.
3. Randomly generate an object of $C$ of size $n$.
4. Establish transformations between $C$ and other combinatorial collections $D$. 
Some combinatorial collections

— Words —

Let $A := \{a, b\}$ be an alphabet and let $A^*$ be the combinatorial collection of all words on $A$ where the size of a word is its length.
Some combinatorial collections

— Words —

Let \( A := \{a, b\} \) be an alphabet and let \( A^* \) be the combinatorial collection of all words on \( A \) where the size of a word is its length.

Then, \( A^*(0) = \{\epsilon\} \), \( A^*(1) = \{a, b\} \), and \( A^*(2) = \{aa, ab, ba, bb\} \).
Some combinatorial collections

— Words —

Let $A := \{a, b\}$ be an alphabet and let $A^*$ be the combinatorial collection of all words on $A$ where the size of a word is its length.

Then, $A^*(0) = \{\epsilon\}$, $A^*(1) = \{a, b\}$, and $A^*(2) = \{aa, ab, ba, bb\}$.

— Permutations —

Let $\mathcal{G}$ be the combinatorial collection of all permutations where the size of a permutation is its length as a word.
Some combinatorial collections

— Words —

Let $A := \{a, b\}$ be an alphabet and let $A^*$ be the combinatorial collection of all words on $A$ where the size of a word is its length.

Then, $A^*(0) = \{\epsilon\}$, $A^*(1) = \{a, b\}$, and $A^*(2) = \{aa, ab, ba, bb\}$.

— Permutations —

Let $\mathcal{S}$ be the combinatorial collection of all permutations where the size of a permutation is its length as a word.

Then, $\mathcal{S}(0) = \{\epsilon\}$, $\mathcal{S}(1) = \{1\}$, $\mathcal{S}(2) = \{12, 21\}$, and $\mathcal{S}(3) = \{123, 132, 213, 231, 312, 321\}$.
Some combinatorial collections

— Words —

Let $A := \{a, b\}$ be an alphabet and let $A^*$ be the combinatorial collection of all words on $A$ where the size of a word is its length.

Then, $A^*(0) = \{\epsilon\}$, $A^*(1) = \{a, b\}$, and $A^*(2) = \{aa, ab, ba, bb\}$.

— Permutations —

Let $\mathcal{G}$ be the combinatorial collection of all permutations where the size of a permutation is its length as a word.

Then, $\mathcal{G}(0) = \{\epsilon\}$, $\mathcal{G}(1) = \{1\}$, $\mathcal{G}(2) = \{12, 21\}$, and $\mathcal{G}(3) = \{123, 132, 213, 231, 312, 321\}$.

— Binary trees —

Let $\mathcal{BT}$ be the combinatorial collection of all binary trees where the size of a binary tree is its number of internal nodes.
Some combinatorial collections

— Words —

Let \( A := \{a, b\} \) be an alphabet and let \( A^* \) be the combinatorial collection of all words on \( A \) where the size of a word is its length.

Then, \( A^*(0) = \{\epsilon\} \), \( A^*(1) = \{a, b\} \), and \( A^*(2) = \{aa, ab, ba, bb\} \).

— Permutations —

Let \( \mathcal{G} \) be the combinatorial collection of all permutations where the size of a permutation is its length as a word.

Then, \( \mathcal{G}(0) = \{\epsilon\} \), \( \mathcal{G}(1) = \{1\} \), \( \mathcal{G}(2) = \{12, 21\} \), and \( \mathcal{G}(3) = \{123, 132, 213, 231, 312, 321\} \).

— Binary trees —

Let \( \text{BT} \) be the combinatorial collection of all binary trees where the size of a binary tree is its number of internal nodes.

Then, \( \text{BT}(0) = \{\text{ }\} \), \( \text{BT}(1) = \{\text{ }\} \), \( \text{BT}(2) = \{\text{ }\} \), and

\( \text{BT}(3) = \{\text{ }\} \).
Generating series

The generating series of a combinatorial collection $C$ is

$$G_C(t) := \sum_{n \in \mathbb{N}} \#C(n)t^n = \sum_{x \in C} t^{|x|}. $$

Generating series are very powerful tools for enumeration. They encode sequences of numbers and support many operations.
Generating series

The generating series of a combinatorial collection $C$ is

$$G_C(t) := \sum_{n \in \mathbb{N}} \#C(n)t^n = \sum_{x \in C} t^{|x|}.$$ 

— Examples —

▶ $G_{A^*}(t) = 1 + 2t + 4t^2 + 8t^3 + 16t^4 + 32t^5 + \cdots = \frac{1}{1 - 2t}$

Generating series are very powerful tools for enumeration. They encode sequences of numbers and support many operations.
Generating series

The generating series of a combinatorial collection $C$ is

$$G_C(t) := \sum_{n\in\mathbb{N}} \#C(n)t^n = \sum_{x\in C} t^{|x|}.$$  

— Examples —

$\begin{align*}
\Rightarrow G_{A^*}(t) &= 1 + 2t + 4t^2 + 8t^3 + 16t^4 + 32t^5 + \cdots = \frac{1}{1 - 2t} \\
\Rightarrow G_{\bigcirc}(t) &= 1 + t + 2t^2 + 6t^3 + 24t^4 + 120t^5 + \cdots = \int_0^\infty \frac{\exp(-x)}{1 - xt} \, dx
\end{align*}$

Generating series are very powerful tools for enumeration. They encode sequences of numbers and support many operations.
Generating series

The generating series of a combinatorial collection \( C \) is

\[
G_C(t) := \sum_{n \in \mathbb{N}} \#C(n)t^n = \sum_{x \in C} t^{|x|}.
\]

— Examples —

\[
\begin{align*}
\text{▶ } G_{A^*}(t) &= 1 + 2t + 4t^2 + 8t^3 + 16t^4 + 32t^5 + \cdots = \frac{1}{1 - 2t} \\
\text{▶ } G_{\boxtimes}(t) &= 1 + t + 2t^2 + 6t^3 + 24t^4 + 120t^5 + \cdots = \int_0^\infty \frac{\exp(-x)}{1 - xt} \, dx \\
\text{▶ } G_{BT}(t) &= 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + \cdots = \frac{1 - \sqrt{1 - 4t}}{2t}
\end{align*}
\]

Generating series are very powerful tools for enumeration. They encode sequences of numbers and support many operations.
Outline

Algebraic combinatorics
Endow $C$ with operations to form an algebraic structure.
Operations and algebraic structures

— First idea —

Endow $C$ with operations to form an algebraic structure.

The algebraic study of $C$ helps to discover combinatorial properties.
Operations and algebraic structures

— First idea —

Endow $C$ with operations to form an algebraic structure.

The algebraic study of $C$ helps to discover combinatorial properties.

In particular,

1. minimal generating families of $C$

   $\leadsto$ highlighting of elementary pieces of assembly;
Operations and algebraic structures

— First idea —

Endow $C$ with operations to form an algebraic structure.

The algebraic study of $C$ helps to discover combinatorial properties.

In particular,

1. minimal generating families of $C$
   - highlighting of elementary pieces of assembly;

2. morphisms involving $C$
   - transformation algorithms and revelation of symmetries.
Operations and algebraic structures

— First idea —

Endow $C$ with operations to form an algebraic structure.

The algebraic study of $C$ helps to discover combinatorial properties.

In particular,

1. minimal generating families of $C$
   $\mapsto$ highlighting of elementary pieces of assembly;

2. morphisms involving $C$
   $\mapsto$ transformation algorithms and revelation of symmetries.

Most important algebraic structures are

- lattices;
- monoids;
- Hopf bialgebras;
- operads.
Formal power series

Generating series forget a lot of information about the underlying combinatorial objects of $C$. 
Formal power series

Generating series forget a lot of information about the underlying combinatorial objects of $C$.

— Second idea —

Work with formal series of combinatorial objects of $C$. 
Formal power series

Generating series forget a lot of information about the underlying combinatorial objects of $C$.

— Second idea —

Work with formal series of combinatorial objects of $C$.

— Example —

We work with the formal power series wherein exponents are combinatorial objects:

$$f_{BT} = t^1 + t^2 + t^3 + t^4 + t^5 + t^6 + t^7 + t^8 + \cdots$$

instead of the generating series $G_{BT}(t)$. 
Formal power series

Generating series forget a lot of information about the underlying combinatorial objects of $C$.

— Second idea —

Work with formal series of combinatorial objects of $C$.

— Example —

We work with the formal power series wherein exponents are combinatorial objects:

$$ f_{BT} = t^1 + t^2 + t^3 + t^4 + t^5 + t^6 + t^7 + t^8 + \cdots $$

instead of the generating series $G_{BT}(t)$.

If $C$ is endowed with operations $\star$, these operations extend as products $\bar{\star}$ on formal power series leading to expressions for $f_C$.
Outline

Operads and graded graphs
Endowing a combinatorial collection $C$ with the structure of an operad consists in providing a map

$$\circ_i : C(n) \times C(m) \to C(n + m - 1), \quad 1 \leq i \leq n, \ 1 \leq m,$$

satisfying some axioms.
Operad structures

Endowing a combinatorial collection $C$ with the structure of an operad consists in providing a map

$$\circ_i : C(n) \times C(m) \to C(n + m - 1), \quad 1 \leq i \leq n, \ 1 \leq m,$$

satisfying some axioms.

Intuition: for any $x, y \in C$ and $i \in [|x|]$, $x \circ_i y$ can be thought as the insertion of $y$ into the $i$th substitution place of $x$. For instance,
Some operads

— Operad on words —

Let $A := \mathbb{Z}/\ell\mathbb{Z}$ be an alphabet. We turn $A^*$ into an operad where $u \circ_i v$ is obtained by replacing the $i$th letter of $u$ by a copy of $v$ obtained by incrementing $(\text{mod } \ell)$ its letters by $u_i$ [Giraudo, 2015]. For instance, for $\ell := 3$,

$$100210 \circ_5 1022 = 100221000.$$
Some operads

— Operad on words —

Let $A := \mathbb{Z}/\ell\mathbb{Z}$ be an alphabet. We turn $A^*$ into an operad where $u \circ_i v$ is obtained by replacing the $i$th letter of $u$ by a copy of $v$ obtained by incrementing (mod $\ell$) its letters by $u_i$ [Giraudo, 2015]. For instance, for $\ell := 3$,

$$100210 \circ_5 1022 = 100221000.$$ 

— Operad on permutations —

We turn $\mathcal{S}$ into an operad where $\sigma \circ_i \nu$ is the permutation whose permutation matrix is obtained by replacing the $i$th point of the matrix of $\sigma$ by a copy of the matrix of $\nu$ [Aguiar, Livernet, 2007]. For instance,

$$35412 \circ_3 132 = 3746512,$$

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{perm_operad}
\end{figure}
Some operads

Operad on trees

Let $G$ be a set of nodes. We turn the set of trees on $G$ into an operad $F(G)$ where $t \circ_i s$ is obtained by grafting the root of a copy of $s$ onto the $i$th leaf of $t$. For instance, for $G := \{\bullet, \circ, \bullet\}$, we have

$$o_5 \quad \quad = \quad \quad .$$
Some operads

— Operad on trees —

Let $G$ be a set of nodes. We turn the set of trees on $G$ into an operad $\mathbf{F}(G)$ where $t \circ_i s$ is obtained by grafting the root of a copy of $s$ onto the $i$th leaf of $t$. For instance, for $G := \{ \bullet, \circ, \circ \}$, we have

There exist many other (more or less complicated) operads involving combinatorial objects:

- on various families of trees (binary trees, $m$-trees, Schröder trees, rooted trees, etc.);
- on various families of paths (Dyck paths, Motzkin paths, etc.);
- on various families of graphs (cliques, drawn inside a polygon, with labeled edges, etc.).
Young lattice

One of the most famous graded graphs is the Hasse diagram of the Young lattice.
Young lattice

One of the most famous graded graphs is the Hasse diagram of the Young lattice.

The vertices of this graph are integer partitions, nonincreasing words of positive integers.
Young lattice

One of the most famous graded graphs is the Hasse diagram of the Young lattice.

The vertices of this graph are integer partitions, nonincreasing words of positive integers.

The Young lattice admits as Hasse diagram the graph wherein there is an arc $\lambda \rightarrow \mu$ if $\mu$ can be obtained by adding a box from $\lambda$: 

![Diagram of the Young lattice]
Graded graphs

A graded graph is a pair $(C, U)$ where $C$ is a combinatorial collection and $U$ is a linear map

$$U : \mathbb{K} \langle C(d) \rangle \to \mathbb{K} \langle C(d + 1) \rangle, \quad d \geq 0.$$ 

This map sends any $x \in C$ to its next vertices (with multiplicities).
A graded graph is a pair \((C, U)\) where \(C\) is a combinatorial collection and \(U\) is a linear map

\[ U : \mathbb{K} \langle C(d) \rangle \rightarrow \mathbb{K} \langle C(d + 1) \rangle, \quad d \geq 0. \]

This map sends any \(x \in C\) to its next vertices (with multiplicities).

Classical examples include

- the Young lattice [Stanley, 1988];
- the bracket tree [Fomin, 1994];
- the composition poset [Björner, Stanley, 2005];
- the Fibonacci lattice [Fomin, 1988], [Stanley, 1988].
Graded graphs and duality

These graphs become very interesting if we consider two such structures \((C, U)\) and \((C, V)\) at the same time, sharing the same underlying set \(C\).
Graded graphs and duality

These graphs become very interesting if we consider two such structures \((C, U)\) and \((C, V)\) at the same time, sharing the same underlying set \(C\).

We look for the following properties:

- **duality** \cite{Stanley} if

\[
V^* U - UV^* = I;
\]

- **\(r\)-duality** \cite{Fomin} if

\[
V^* U - UV^* = rI
\]

for an \(r \in K\);

- **\(\phi\)-diagonal duality** \cite{Giraudo} if

\[
V^* U - UV^* = \phi
\]

for a nonzero diagonal linear map \((\phi(x) = \lambda x)\) where \(\lambda x \in K\) \{0\}).
Graded graphs and duality

These graphs become very interesting if we consider two such structures \((
C, U)\) and \((C, V)\) at the same time, sharing the same underlying set \(C\).

We look for the following properties:

► duality [Stanley, 1988] if

\[
V^*U - UV^* = I;
\]

► \(r\)-duality [Fomin, 1994] if

\[
V^*U - UV^* = rI
\]

for an \(r \in K\).
Graded graphs and duality

These graphs become very interesting if we consider two such structures \((\mathcal{C}, \mathcal{U})\) and \((\mathcal{C}, \mathcal{V})\) at the same time, sharing the same underlying set \(\mathcal{C}\).

We look for the following properties:

- **duality** [Stanley, 1988] if
  \[
  V^*U - UV^* = I;
  \]

- **\(r\)-duality** [Fomin, 1994] if
  \[
  V^*U - UV^* = rI
  \]
  for an \(r \in \mathbb{K}\);

- **\(\phi\)-diagonal duality** [Giraudo, 2018] if
  \[
  V^*U - UV^* = \phi
  \]
  for a nonzero diagonal linear map \((\phi(x) = \lambda_x x\) where \(\lambda_x \in \mathbb{K} \setminus \{0\}\).
Graded graphs and duality

These graphs become very interesting if we consider two such structures \((C, U)\) and \((C, V)\) at the same time, sharing the same underlying set \(C\).

We look for the following properties:

- **duality** [Stanley, 1988] if
  \[ V^*U - UV^* = I; \]

- **\(r\)-duality** [Fomin, 1994] if
  \[ V^*U - UV^* = rI \]
  for an \(r \in \mathbb{K};\)

- **\(\phi\)-diagonal duality** [Giraudo, 2018] if
  \[ V^*U - UV^* = \phi \]
  for a nonzero diagonal linear map \((\phi(x) = \lambda_x x \text{ where } \lambda_x \in \mathbb{K} \setminus \{0\}).\)

---

**Idea**

Use operads as a source of dual pairs of graded graphs.
Graded graphs from operads

— Example —

For $G = \{ \bullet, \circ \}$, the pair $(F(G), U, V)$ is
Graded graphs from operads

— Example —

For $G = \{\circ, \circ\}$, the pair $(F(G), U, V)$ is

\[ U(x) := \sum_{a \in G} \sum_{i \in [|x|]} x \circ_i a, \quad V(x) := \sum_{y \in \mathcal{O}} y \quad \text{such that} \quad \exists (s, t) \in \text{ev}^{-1}(x) \times \text{ev}^{-1}(y) \quad \langle t, V(s) \rangle \neq 0. \]

General construction: given an operad $\mathcal{O}$ (satisfying some conditions), let the graphs $(\mathcal{O}, U)$ and $(\mathcal{O}, V)$ defined by
Graded graphs from operads

— Theorem [Giraudo, 2018] —

If $\mathcal{O}$ is an homogeneous operad, then $(\mathcal{O}, U, V)$ is a pair of graded graphs. Moreover, if $\mathcal{O}$ is a free operad, this pair is \(\phi\)-diagonal dual.
Graded graphs from operads

— Theorem [Giraudo, 2018] —

If $\mathcal{O}$ is an homogeneous operad, then $(\mathcal{O}, U, V)$ is a pair of graded graphs. Moreover, if $\mathcal{O}$ is a free operad, this pair is $\phi$-diagonal dual.

There are non-free operads leading to $\phi$-diagonal dual graphs.
Graded graphs from operads

— Theorem [Giraudo, 2018] —

If $O$ is an homogeneous operad, then $(O, U, V)$ is a pair of graded graphs. Moreover, if $O$ is a free operad, this pair is $\phi$-diagonal dual.

There are non-free operads leading to $\phi$-diagonal dual graphs.

— Example —

The pair $(\text{Comp}, U, V)$ is 2-dual. The graded graph $(\text{Comp}, U)$ is the Hasse diagram of the composition poset [Bjöner, Stanley, 2005].
Graded graphs from operads

--- Theorem [Giraudo, 2018] ---

If $\mathcal{O}$ is an homogeneous operad, then $(\mathcal{O}, U, V)$ is a pair of graded graphs. Moreover, if $\mathcal{O}$ is a free operad, this pair is $\phi$-diagonal dual.

There are non-free operads leading to $\phi$-diagonal dual graphs.

--- Example ---

The pair $(\text{Comp}, U, V)$ is 2-dual. The graded graph $(\text{Comp}, U)$ is the Hasse diagram of the composition poset [Björner, Stanley, 2005].

--- Example ---

The pair $(\text{Motz}, U, V)$ is $\phi$-diagonal dual.
Further reading on operads

